Reliable Networks With Unreliable Sensors*

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Abstract. Wireless sensor networks (WSNs) deployed in hostile environments suffer from a high rate of node failure. We investigate the effect of such failure rate on network connectivity. We provide a formal analysis that establishes the relationship between node density, network size, failure probability, and network connectivity. We show that as network size and density increase, the probability of network partitioning becomes arbitrarily small. We show that large networks can maintain connectivity despite a significantly high probability of node failure. We derive mathematical functions that provide lower bounds on network connectivity in WSNs. We compute these functions for some realistic values of node reliability, area covered by the network, and node density, to show that, for instance, networks with over a million nodes can maintain connectivity with a probability exceeding 99% despite node failure probability exceeding 57%.

1 Introduction

Wireless Sensor Networks (WSNs) [2] are being used in a variety of applications ranging from volcanology [21] and habitat monitoring [18] to military surveillance [10]. Often, in such deployments, premature uncontrolled node crashes are common. The reasons for this include, but are not limited to, hostility of the environment (like extreme temperature, humidity, soil acidity, and such), node fragility (especially if the nodes are deployed from the air on to the ground), and the quality control in the manufacturing of the sensors. Consequently, crash fault tolerance becomes a necessity (not just a desirable feature) in WSNs. Typically, a sufficiently dense node distribution with redundancy in connectivity and coverage provides the necessary fault tolerance. In this paper, we analyze the connectivity fault tolerance of such large scale sensor networks and show how, despite high unreliability, flaky sensors can build robust networks.

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The results in this paper address the following questions: Given a static WSN deployment (of up to a few million nodes) where (a) the node density is D nodes per unit area, (b) the area of the region is Z units, and (c) each node can fail[3] with an independent and uniform probability ρ: what is the probability P that the network is connected (that is, the network is not partitioned)? What is the relationship between P, ρ, D, and Z?

Motivation. The foregoing questions are of significant practical interest. A typical specification for designing a WSN is the area of coverage, an upper bound on the (financial) cost, and QoS guarantees on connectivity (and coverage). High reliability sensor nodes offer better guarantees on connectivity but also increase the cost. An alternative is to reduce the costs by using less reliable nodes, but the requisite guarantees on connectivity might necessitate greater node density (that is, greater number of nodes per unit area), which again increases the cost. As a network designer, it is desirable to have a function that accepts, as input, the specifications of a WSN and outputs feasible and appropriate design choices. We derive the elements of such a function in Sect. 6 and demonstrate the use of the results from Sect. 6 in Sect. 7.

Contribution. This paper has three main contributions. First, we formalize and prove the intuitive conjecture that as node reliability and/or node density of a WSN increases, the probability of connectivity also increases. We provide a probabilistic analysis for the relationship between node reliability (ρ), node density (D), area of the WSN region (Z), and the probability of network connectivity(P); we provide lower bounds for P as a function of ρ, D, and Z.

Second, we provide concrete lower bounds for expected connectivity probability for various reasonable values of ρ, D, and Z.

Third, we use a new technique of hierarchical network analysis to derive the lower bounds on a non-hierarchical WSN. To our knowledge, we are the first to utilize this approach in wireless sensor networks. The approach, model, and proof techniques themselves may be of independent interest.

Organization. The rest of this paper is organized as follows: The related work is described next in Section 2. The system model assumptions are discussed in Section 3. The methodology includes tiling the plane with regular hexagons. The analysis and results in this paper use a topological object called a *level-z polyhex* that is derived from a regular hexagon. The level-z polyhex is introduced in Section 4. Section 5 introduces the notion of *level-z connectedness* of an arbitrary WSN region. Section 6 uses this notion of level-z to formally establish the relationship between P, ρ, D, and Z. Finally, section 7 provides lower bounds on connectivity for various values of ρ, D, and Z.

2 Related Work

There is a significant body of work on static analysis of topological issues associated with WSNs [12]. These issues are discussed in the context of coverage [13], connectivity [19], and routing [1].

[3] Node is said to fail if it crashes prior to its intended lifetime. See Sect. 3 for details.

The results in [19] focus on characterizing the fault tolerance of sensor networks by establishing the k-connectivity of a WSN. However, such characterization results in a poor lower bound of $k-1$ on the fault tolerance which corresponds to the worst-case behavior of faults. It fails to characterize the expected probability of network partitioning in practical deployments. In other related results, Bhandari *et al.* [5] focus on optimal node density (or degree) for a WSN to be connected w.h.p, and Kim *et al.* [11] consider connectivity in randomly duty-cycled WSNs in which nodes take turns to be active to conserve power. A variant of network connectivity, called partial connectivity, is explored in [6] which derives derives the relationship between node density and the percentage f of the network expected to be connected. Our research addresses a different, but related question: given a fixed WSN region with a fixed initial node density (and hence, degree) and a fixed failure probability, what is the probability that the WSN will remain connected?

The results in [16, 4, 22, 20, 3] establish and explore the relationship between coverage and connectivity. The results in [22] and [20] show that in large sensor networks if the communication radius r_c is at least twice the coverage radius r_s, then coverage of a convex area implies connectivity among the non-faulty nodes. In [4], Bai *et al.* establish optimal coverage and connectivity in regular patterns including square grids and hexagonal lattice where $r_c/r_s < 2$ by deploying additional sensors at specific locations. Results from [16] show that even if $r_c = r_s$, large networks in a square region can maintain connectivity despite high failure probability; however, connectivity does not imply coverage. Ammari *et al.*, extend these results in [3] to show that if $r_c/r_s = 1$ in a k-covered WSN, then the network fault tolerance is given by $4r_c(r_c + r_s)k/r_s^2 - 1$ for a sparse distribution of node crashes. Another related result [17] shows that in a uniform random deployment of sensors in a WSN covering the entire region, the probability of maintaining connectivity approaches 1 as r_c/r_s approaches 2.

Our work differs from the works cited above in three aspects: (a) we focus exclusively on maintaining total connectivity, (b) while the results in [16, 4, 22, 20] apply to specific deployment patterns or shape of a region, our results and methodology can be applied to any arbitrary region and any constant node density, and (c) our analysis is probabilistic insofar as node crashes are assumed to be independent random events, and we focus on the probability of network connectivity in the average case instead of the worst case.

The tiling used in our model induces a hierarchical structure which can be used to decompose the connectivity property of a large network into connectivity properties of constituent smaller sub-networks of similar structure. This approach was first introduced in [9], and subsequently used to analyze fault tolerance of hypercube networks [7] and mesh networks [8]. Our approach differs from those in [7] and [8] as we construct higher order polyhex tiling using the underlying hexagons to derive a recursive function that establishes a lower bound on network connectivity as a function of ρ and D.

3 System Model

We make the following simplifying assumptions:

- **Node.** The WSN has a finite fixed set of n nodes. Each node has a communication radius R.
- **Region and tiles.** A WSN region is assumed to be a finite plane tiled by regular hexagons whose sides are of length l such that nodes located in a given hexagon can communicate reliably[4] with all the nodes in the same hexagon and adjacent hexagons. We assume that each hexagon contains at least D nodes.
- **Faults.** A node can fail only by crashing before the end of its intended lifetime. Faults are independent and each node has a constant probability ρ of failing.
- **Empty tile.** A hexagon is said to be *empty* if it contains only faulty nodes.

We say that two non-faulty nodes p and p' are *connected* if either p and p' are in the same or neighboring hexagons, or there exists some sequence of non-faulty nodes $p_i, p_{i+1}, \ldots, p_j$ such that p (and p', respectively) and p_i (and p_j, respectively) are in adjacent hexagons, and p_k and p_{k+1} are in adjacent hexagons, where $i \leq k \leq j$. We say that a region is *connected* if every pair of non-faulty nodes p and p' in the region are connected.

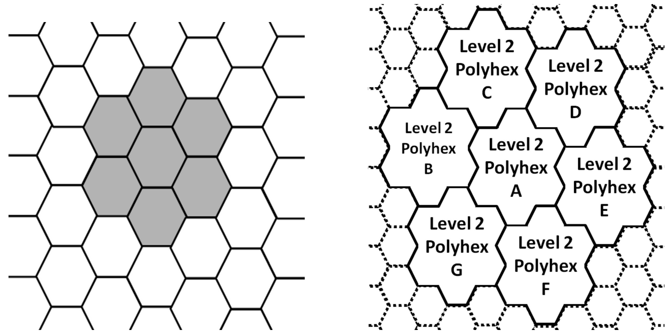
4 Higher Level Tilings: Polyhexes

For the analysis of WSNs in an arbitrary region, we use of the notion of higher level tilings by grouping sets of contiguous hexagons into 'super tiles' such that some specific properties (like the ability to tile the Euclidean plane) are preserved. Such 'super tiles' are called level-z polyhexes. Different values of z specify different level-z polyhexes. In this section we define a level-z polyhex and specify its properties.

The following definitions are borrowed from [14]: A *tiling* of the Euclidean plane is a countable family of closed sets called tiles, such that the union of the sets is the entire plane and such that the interiors of the sets are pairwise disjoint. We are concerned only with *monohedral* tilings — tilings in which every tile is congruent to a single fixed tile called the *prototile*. In our case, a regular hexagon is a prototile. We say that the prototile *admits* the tiling. A *patch* is a finite collection of non-overlapping tiles such that their union is a closed topological disk[5]. A *translational patch* is a patch such that the tiling consists entirely of a lattice of translations of that patch.

[4] We assume that collision resolution techniques are always successful in ensuring reliable communication.

[5] A closed topological disk is the image of a closed circular disk under a homeomorphism. Roughly speaking, homeomorphism is a continuous stretching and bending of the object into a new shape (you are not allowed to tear or 'cut holes' into the object). Thus, any two-dimensional shape that has a closed boundary, finite area, and no 'holes' is a closed topological disk. This includes squares, circles, ellipses, hexagons, and polyhexes.



(a) The gray tiles form a level-2 polyhex.

(b) A level-3 polyhex formed by 7 level-2 polyhexes A–F.

Fig. 1. Examples of Polyhexes

We now define a translational patch of regular hexagons called *level-z polyhexes* for $z \in \mathbb{N}$ as follows:

– A level-1 polyhex is a regular hexagon: a prototile.
– A level-z polyhex for $z > 1$ is a translational patch of seven level-$(z-1)$ polyhexes that admits a hexagonal tiling.

Note that each level-z polyhex is made of seven level-$(z-1)$ polyhexes. Therefore, the total number of tiles in a level-z polyhex is $size(z) = 7^{z-1}$.

Figure 1(a) illustrates the formation of a level-2 polyhex with seven regular hexagons, and Fig. 1(b) illustrates how seven level-2 polyhexes form a level-3 polyhex. A formal proof that such level-z polyhexes exist for arbitrary values of z (in an infinite plane tessellated by regular hexagons) is available at [15].

5 Level-z Polyhexes and Connectivity

The analysis in Section 6 is based on the notion of *level-z connectedness* that is introduced here. First, we define a 'side' to each level-z polyhex. Second, we introduce the concepts of *connected level-z polyhexes* and *level-z connectedness* in a WSN region. Finally, we show how level-z connectedness implies that all non-faulty nodes in a level-z polyhex of a WSN are connected. We use this result and the definition of level-z connectedness to derive a lower bound on the probability of network connectivity in Section 6.

Side. The set of boundary hexagons that are adjacent to a given level-z polyhex are said be a 'side' of the level-z polyhex. Since a level-z polyhex can have 6 neighboring level-z polyhexes, every level-z polyhex has 6 'sides'. The number of hexagons along each 'side' (also called the 'length of the side') is given by $sidelen(z) = 1 + \sum_{i=0}^{z-2} 3^i$ where $z \geq 2$.[6]

We now define what it means for a level-z polyhex to be connected. Intuitively, we say that a level-z polyhex is *connected* if the network of nodes in the level-z polyhex is not partitioned.

Connected level-z polyhex. A level-z polyhex T_{zi} is said to be *connected* if, given the set Λ of all hexagons in T_{zi} that contain at least one non-faulty node, for every pair of hexagons p and q from Λ, there exists some (possibly empty) sequence of hexagons $t_1, t_2, \ldots, t_j$ such that $\{t_1, t_2, \ldots, t_j\} \subseteq \Lambda$, and t_1 is a neighbor of p, every t_i is a neighbor of t_{i+1}, and t_j is a neighbor of q.

Note that if a level-z polyhex is *connected*, then all the non-faulty nodes in the level-z polyhex are connected as well.

We are now ready to define the notion of *level-z connectedness* in a WSN region.

Level-z connectedness. A WSN region $\mathcal{W}$ is said to be level-z connected if there exists some partitioning of $\mathcal{W}$ into disjoint level-z polyhexes such that each such level-z polyhex is *connected*, and for every pair of such level-z polyhexes T_{zp} and T_{zq}, there exists some (possibly empty) sequence of (connected) level-z polyhexes $T_{z1}, T_{z2}, \ldots, T_{zj}$ (from the partitioning of $\mathcal{W}$) such that T_{z1} is a neighbor of T_{zp}, every T_{zi} is a neighbor of $T_{z(i+1)}$, and T_{zj} is a neighbor of T_{zq}. Additionally, each 'side' of T_{zi} has at least $\lceil \frac{sidelen(z)}{2} \rceil$ non-empty hexagons.

We are now ready to prove the following theorem:

Theorem 1 *Given a WSN region $\mathcal{W}$, if $\mathcal{W}$ is level-z connected, then all non-faulty nodes in $\mathcal{W}$ are connected.*

Proof. Suppose that the region $\mathcal{W}$ is level-z connected. It follows that there exists some partitioning Λ of $\mathcal{W}$ into disjoint level-z polyhexes such that each such level-z polyhex is connected, and for every pair of such level-z polyhexes T_{zp} and T_{zq}, there exists some (possibly empty) sequence of (connected) level-z polyhexes $T_{z1}, T_{z2}, \ldots, T_{zj}$ (from the partitioning of $\mathcal{W}$) such that T_{z1} is a neighbor of T_{zp}, every T_{zi} is a neighbor of $T_{z(i+1)}$, and T_{zj} is a neighbor of T_{zq}. Additionally, each 'side' of T_{zi} has at least $\lceil \frac{sidelen(z)}{2} \rceil$ non-empty hexagons.

To prove the theorem, it is sufficient to show that for any two non-faulty nodes in $\mathcal{W}$ in hexagons p and q, respectively, the hexagons p and q are connected.

Let hexagon p lie in a level-z polyhex T_{zp} ($\in \Lambda$), and let q lie in a level-z polyhex T_{zq} ($\in \Lambda$). Note that since Λ is a partitioning of $\mathcal{W}$, either $T_{zp} = T_{zq}$ or T_{zp} and T_{zq} are disjoint. If $T_{zp} = T_{zq}$, then since T_{zp} is *connected*, it follows that p and q are connected. Hence, all non-faulty nodes in p are *connected* with all non-faulty nodes in q. Thus, the theorem is satisfied.

If T_{zp} and T_{zq} are disjoint, then it follows from the definition of level-z connectedness that there exists some sequence of *connected* level-z polyhex $T_{z1}, T_{z2}, \ldots, T_{zj}$ such that T_{z1} is a neighbor of T_{zp}, every T_{zi} is a neighbor of

[6] The proof for this equation is a straightforward induction on z and the proof has been omitted.

$T_{z(i+1)}$, and T_{zj} is a neighbor of T_{zq}. Additionally, each 'side' of T_{zi} has at least $\lceil \frac{sidelen(z)}{2} \rceil$ non-empty hexagons.

Consider any two neighboring level-z polyhexes $(T_{zm}, T_{zn}) \in \Lambda \times \Lambda$. Each 'side' of T_{zm} and T_{zn} has $sidelen(z)$ hexagons. Therefore, T_{zm} and T_{zn} have $sidelen(z)$ boundary hexagons such that each such hexagon from T_{zm} (and respectively, T_{zn}) is adjacent to two boundary hexagons in T_{zn} (and respectively, T_{zm}), except for the two boundary hexagons on either end of the 'side' of T_{zm} (and respectively, T_{zn}); these two hexagons are adjacent to just one hexagon in T_{zn} (and respectively, T_{zm}). We know that at least $\lceil \frac{sidelen(z)}{2} \rceil$ of these boundary hexagons are non-empty. It follows that there exists at least one non-empty hexagon in T_{zm} that is adjacent to a non-empty hexagon in T_{zn}. Such a pair of non-empty hexagons (one in T_{zm} and the other in T_{zn}) form a "bridge" between T_{zm} and T_{zn} allowing nodes in T_{zm} to communicate with nodes in T_{zn}. Since T_{zm} and T_{zn} are *connected* level-z polyhexes, it follows that nodes within T_{zm} and T_{zn} are *connected* as well. Additionally, we have established that there exist at least two hexagons, one in T_{zm} and one in T_{zn} that are *connected.* It follows that nodes in T_{zm} and T_{zn} are *connected* with each other as well.

Thus, it follows that T_{zp} and T_{z1} are connected, every T_{zi} is connected with $T_{z(i+1)}$, and T_{zj} is connected with T_{zq}. From the transitivity of connectedness, it follows that T_{zp} is connected with T_{zq}. That is, all non-faulty nodes in hexagon p are connected with all non-faulty nodes in q. Since p and q are arbitrary hexagons in $\mathcal{W}$, it follows that all the nodes in $\mathcal{W}$ are connected.

Theorem 1 provides the following insight into connectivity analysis of a WSN: for appropriate values of z, a level-z polyhex has fewer nodes than the entire region $\mathcal{W}$. In fact, a level-z polyhex could have orders of magnitude fewer nodes than $\mathcal{W}$. Consequently, the analysis of connectedness of a level-z polyhex is simpler and easier than the connectedness of the entire region $\mathcal{W}$. Using Theorem 1, we can leverage such an analysis of a level-z polyhex to derive a lower bound on the connectivity probability of $\mathcal{W}$. The foregoing motivation is explored next.

6 On Fault Tolerance of WSN Regions

We are now ready to derive a lower bound on the connectivity probability of an arbitrarily-shaped WSN region. Let $\mathcal{W}$ be a WSN region with node density of D nodes per hexagon such that the region is approximated by a patch of x level-z polyhexes that constitute a set Λ. Let each node in the region fail independently with probability ρ. Let $Conn_{\mathcal{W}}$ denote the event that all the non-faulty nodes in the region $\mathcal{W}$ are connected. Let $Conn_{(T,z,side)}$ denote the event that a level-z polyhex T is connected and each 'side' of T has at least $\lceil sidelen(z)/2 \rceil$ non-empty hexagons.

We know that if $\mathcal{W}$ is level-z connected, then all the non-faulty nodes in $\mathcal{W}$ are connected. Also, $\mathcal{W}$ is level-z connected if: $\forall T \in \Lambda :: Conn_{(T,z,side)}$. Therefore, the probability that $\mathcal{W}$ is connected is bounded by: $Pr\left[Conn_{\mathcal{W}}\right] \geq (Pr\left[Conn_{(T,z,side)}\right])^x$.

Thus, in order to find a lower bound on $Pr\left[Conn_{\mathcal{W}}\right]$, we have to find the lower bound on $(Pr\left[Conn_{(T,z,side)}\right])^x$.

Lemma 2 *In a level-z polyhex T with node density of D nodes per hexagon, suppose each node fails independently with a probability ρ. Then the probability that T is connected and each 'side' of T has at least $\lceil sidelen(z)/2\rceil$ non-empty hexagons is given by $Pr\left[Conn_{(T,z,side)}\right] = \sum_{i=0}^{size(z)} N_{z,i}(1-\rho^D)^{size(z)-i}\rho^{D\times i}$, where $N_{z,i}$ is the number of ways by which we can have i empty hexagons and $size(z)-i$ non-empty hexagons in a level-z polyhex such that the level-z polyhex is connected and each 'side' of the level-z polyhex has at least $\lceil sidelen(k)/2\rceil$ non-empty hexagons.*

Proof. Fix i hexagons in T to be empty such that T is connected and each 'side' of T has at least $\lceil sidelen(k)/2\rceil$ non-empty hexagons. Since nodes fail independently with probability ρ, and there are D nodes per hexagon, the probability that a hexagon is empty is ρ^D. Therefore, the probability that exactly i hexagons are empty in T is given by $(1-\rho^D)^{size(z)-i}\rho^{D\times i}$. By assumption, there are $N_{z,i}$ ways to fix i hexagons to be empty. Therefore, the probability that T is connected and each 'side' of T has at least $\lceil sidelen(k)/2\rceil$ non-empty hexagons despite i empty hexagons is given by $N_{z,i}(1-\rho^D)^{size(z)-i}\rho^{D\times i}$. However, note that we can set i (the number of empty hexagons) to be anything from 0 to $size(z)$. Therefore, $Pr\left[Conn_{(T,z,side)}\right]$ is given by $\sum_{i=0}^{size(z)} N_{z,i}(1-\rho^D)^{size(z)-i}\rho^{D\times i}$.

Given the probability of $Conn_{(T,z,side)}$, we can now establish a lower bound for the probability that the region $\mathcal{W}$ is connected.

Theorem 3 *Suppose each node in a WSN region $\mathcal{W}$ fails independently with probability ρ, $\mathcal{W}$ has a node density of D nodes per hexagon and tiled by a patch of x level-z polyhexes. Then the probability that all non-faulty nodes in $\mathcal{W}$ are connected is at least $(Pr\left[Conn_{(T,z,side)}\right])^x$*

Proof. There are x level-z polyhexes in $\mathcal{W}$. Note that if $\mathcal{W}$ is level-z connected, then all non-faulty nodes in $\mathcal{W}$ are connected. However, observe that $\mathcal{W}$ is level-z connected if each such level-z polyhex is connected and each 'side' of each such level-z polyhex has at least $\lceil sidelen(z)/2\rceil$ non-empty hexagons. Recall from Lemma 2 that the probability of such an event for each polyhex is given by $Pr\left[Conn_{(T,z,side)}\right]$. Since there are x such level-z polyhex, and failure probability of nodes (and hence disjoint level-z polyhexes) is independent, it follows that the probability of $\mathcal{W}$ being connected is at least $(Pr\left[Conn_{(T,z,side)}\right])^x$.

Note that the lower bound we have established depends on the function $N_{z,i}$ defined in Lemma 2. Unfortunately, to the best of our knowledge, there is no known algorithm that computes $N_{z,i}$ in a reasonable amount of time. Since this is a potentially infeasible approach for large WSNs with millions of nodes, we provide an alternate lower bound for $Pr\left[Conn_{(T,z,side)}\right]$.

Lemma 4 *The value of $Pr\left[Conn_{(T,z,side)}\right]$ from Lemma 2 is bounded below by: $Pr\left[Conn_{(T,z,side)}\right] \geq (Pr\left[Conn_{(T,z-1,side)}\right])^7 + (Pr\left[Conn_{(T,z-1,side)}\right])^6 \times \rho^{D\times size(z-1)}$ where $Pr\left[Conn_{(T,1,side)}\right] = 1-\rho^D$.*

Proof. Recall that a level-z polyhex consists for seven level-$(z-1)$ polyhexes with one internal level-$(z-1)$ polyhex and six outer level-$(z-1)$ polyhexes. Observe that a level-z polyhex satisfies $Conn_{(T,z,side)}$ if either (a) all the seven level-$(z-1)$ polyhexes satisfy $Conn_{(T,z-1,side)}$, or (b) the internal level-$(z-1)$ polyhex is empty and the six outer level-$(z-1)$ polyhexes satisfy $Conn_{(T,z-1,side)}$. From Lemma 2 we know that the probability of a level-$(z-1)$ polyhex satisfying $Conn_{(T,z-1,side)}$ is given by $Pr\left[Conn_{(T,z-1,side)}\right]$ and the probability of a level-$(z-1)$ polyhex being empty is $\rho^{D\times size(z-1)}$. For a level-1 polyhex (which is a regular hexagon tile), the probability that the hexagon is not empty is $1-\rho^D$. Therefore, the probability that cases (a) or (b) is satisfied for $z>1$ is given by $(Pr\left[Conn_{(T,z-1,side)}\right])^7 + (Pr\left[Conn_{(T,z-1,side)}\right])^6 \times \rho^{D\times size(z-1)}$. Therefore, $Pr\left[Conn_{(T,z,side)}\right] \geq (Pr\left[Conn_{(T,z-1,side)}\right])^7 + (Pr\left[Conn_{(T,z-1,side)}\right])^6 \times \rho^{D\times size(z-1)}$ where $Pr\left[Conn_{(T,1,side)}\right] = 1-\rho^D$.

Analyzing the connectivity probability for WSN regions that are level-z connected where z is large, can be simplified by invoking Lemma 4, and reducing the complexity of the computation to smaller values of z for which $Pr\left[Conn_{(T,z,side)}\right]$ can be computed (by brute force) fairly quickly.

7 Discussion

Choosing the size of the hexagon. For the results from the previous section to be of practical use, it is important that we choose the size of the hexagons in our system model carefully. On the one hand, choosing very large hexagons could violate the system model assumption that nodes can communicate with nodes in neighboring hexagons, and on the other hand, choosing small hexagons could result in poor lower bounds and thus result in over-engineered WSNs that incur high costs but with incommensurate benefits.

If we make no assumptions about the locations of nodes within hexagons, then the length l of the sides of a hexagon must be at most $R/\sqrt{13}$ to ensure connectivity between non-faulty nodes in neighboring hexagons. However, if the nodes are "evenly" placed within each hexagon, then l can be as large as $R/2$ while still ensuring connectivity between neighboring hexagons. In both cases, the requirement is that the distance between two non-faulty nodes in neighboring hexagons is at most R.

Computing $N_{z,i}$ from Lemma 2. The function $N_{z,i}$ does not have a closed-form solution. It needs to be computed through exhaustive enumeration. We computed $N_{z,i}$ for some useful values of z and i and included them in Table 1. Using these values, we applied Theorem 3 and Lemma 4 to sensor networks of different sizes, node densities, and node failure probabilities. The results are presented in Table 2. Next, we demonstrate how to interpret and understand the entries in these tables through an illustrative example.

Practicality. Our results can be utilized in the following two practical scenarios. (1) Given an existing WSN with known node failure probability, node density, and area of coverage, we can estimate the probability of connectivity of the

z	i	$N_{z,i}$	z	i	$N_{z,i}$
$k > 2$	1	$size(k) = 7^{k-1}$	3	7	74729943
3	2	1176	3	8	361856172
3	3	18346	3	9	1481515771
3	4	208372	4	2	58653
3	5	1830282	4	3	6666849
3	6	12899198	5	2	2881200

Table 1. Computed Values of $N_{z,i}$

Node density D	No. of Nodes	Node failure prob. ρ	No. of Nodes	Node failure prob. ρ
	$z = 2$ (level-2 polyhex)		$z = 5$ (level-5 polyhex)	
3	21	35%	7203	24%
5	35	53%	12005	40%
10	70	70%	24010	63%
	$z = 3$ (level-3 polyhex)		$z = 6$ (level-6 polyhex)	
3	137	37%	50421	19%
5	245	50%	84035	36%
10	490	70%	24010	63%
	$z = 4$ (level-4 polyhex)		$z = 7$ (level-7 polyhex)	
3	1029	29%	352947	15%
5	1715	47%	588245	31%
10	3430	67%	1176490	57%

Table 2. Various values of node failure probability ρ, node density D, and level-z polyhex that yield network connectivity probability exceeding 99%

entire network. First, we decide on the size of a hexagon as discussed previously, and then we consider level-z polyhexes that cover the region. Next, we apply Theorem 3 and Lemma 4 to compute the probability of connectivity of the network for the given values of ρ, D and z, and the precomputed values of $N_{z,i}$ in Table 1.

(2) The results in this paper can be used to design a network with a specified probability of connectivity. In this case, we decide on a hexagon size that best suits the purposes of the sensor network and determine the level of the polyhex(es) needed to cover the desired area. As an example, consider a 200 sq. km region (approximately circular, so that there are no 'bottle neck' regions) that needs to be covered by a sensor network with a 99% connectivity probability. Let the communication radius of each sensor be 50 meters. The average-case value of the length l of the side of the hexagon is 25 meters, and the 200 sq. km region is tiled by a single level-7 polyhex. From Table 2, we see that if the network consists of 3 nodes per hexagon, then the region will require about 352947 nodes with a failure probability of 15% (85% reliability). However, if the node redundancy

is increased to 5 nodes per hexagon, then the region will require about 588245 nodes with a failure probability of 31% (69% reliability). If the node density is increased further to 10 nodes per hexagon, then the region will require about 1176490 nodes with a failure probability of 57% (43% reliability).

On the lower bounds. An important observation is that these values for node reliability are lower bounds, but are definitely not tight bounds. This is largely because in order to obtain tighter lower bounds, we need to compute the probability of network connectivity from Theorem 3. However, this requires us to compute the values for $N_{z,i}$ for all values of i ranging from 1 to z, which is expensive for z exceeding 3. Consequently, we are forced to use the recursive function in Lemma 4 for computing the network connectivity for larger networks. This reduces the accuracy of the lower bound significantly. A side effect of this error is that in Table 2, we see that for a given D, ρ decreases as z increases. If we were to invest the time and computing resources to compute $N_{z,i}$ for higher values of z (5, 6, 7, and greater), then the computed values for ρ in Table 2 would be significantly larger.

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