Reliable Networks With Unreliable Sensors[☆,☆☆]

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Abstract

Wireless sensor networks (WSNs) deployed in hostile environments suffer from a high rate of node failure. We investigate the effect of such failure rate on network connectivity. We provide a formal analysis that establishes the relationship between node density, network size, failure probability, and network connectivity. We show that large networks can maintain connectivity despite a significantly high probability of node failure. We derive mathematical functions that provide lower bounds on network connectivity in WSNs. We compute these functions for some realistic values of node reliability, area covered by the network, and node density, to show that, for instance, networks with over a million nodes can maintain connectivity with a probability exceeding 95% despite node failure probability exceeding 53%.

Keywords: Sensor network; Connectivity; Fault tolerance

1. Introduction

Wireless Sensor Networks (WSNs) [1] are being used in a variety of applications ranging from volcanology [2] and habitat monitoring [3] to military surveillance [4]. Often, in such deployments, premature uncontrolled node crashes are common. The reasons for this include, but are not limited to, hostility of the environment (like extreme temperature, humidity, soil acidity, and such), node fragility (especially if the nodes are deployed from the air on to the ground),

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and the quality control in the manufacturing of the sensors. Consequently, crash fault tolerance becomes a necessity (not just a desirable feature) in WSNs. Typically, a sufficiently dense node distribution with redundancy in connectivity and coverage provides the necessary fault tolerance. In this paper, we analyze the connectivity fault tolerance of such large scale sensor networks and show how, despite high unreliability, flaky sensors can build robust networks.

The results in this paper address the following questions: Given a static WSN deployment (of up to a few million nodes) where (a) the node density is D nodes per unit area, (b) the area of the region is Z units, and (c) each node can fail[1] with an independent and uniform probability ρ: what is the probability P that the network is connected (that is, the network is not partitioned)? What is the relationship between P, ρ, D, and Z?

Motivation. The foregoing questions are of significant practical interest. A typical specification for designing a WSN is the area of coverage, an upper bound on the (financial) cost, and guarantees on connectivity (and coverage). High reliability sensor nodes offer better guarantees on connectivity but also increase the cost. An alternative is to reduce the costs by using less reliable nodes, but the requisite guarantees on connectivity might necessitate greater node density (that is, greater number of nodes per unit area), which again increases the cost. As a network designer, it is desirable to have a function that accepts, as input, the specifications of a WSN and outputs feasible and appropriate design choices. We derive the elements of such a function and demonstrate their use..

Contribution. This paper has three main contributions. First, we formalize and prove the intuitive conjecture that as node reliability and/or node density of a WSN increases, the probability of connectivity also increases. We provide a probabilistic analysis for the relationship between node reliability (ρ), node density (D), area of the WSN region (Z), and the probability of network connectivity(P); we provide lower bounds for P as a function of ρ, D, and Z.

Second, we provide concrete lower bounds for expected connectivity probability for various reasonable values of ρ, D, and Z.

Third, we use a novel technique for network analysis which, to our knowledge, has not been utilized for wireless sensor networks before. The approach, model, and proof techniques themselves may be of independent interest.

Organization. The rest of this paper is organized as follows: The related work is described next in Section 2. The system model assumptions are discussed in Section 3. The methodology includes tiling the plane with regular hexagons. The analysis and results in this paper use a topological object called a *level-z polyhex* that is derived from a regular hexagon. The level-z polyhex is introduced in Section Appendix A. Section 4 introduces the notion of *level-z connectedness* of an arbitrary WSN region. Section 5 uses this notion of level-z connectedness

[1]Node is said to fail if it crashes prior to its intended lifetime. See Section 3 for details.

to formally establish the relationship between P, ρ, D, and Z. Finally, Section 6 provides lower bounds on connectivity for various values of ρ, D, and Z.

2. Related Work

There is a significant body of work on topological issues associated with WSNs [5]. These issues are discussed in the context of coverage [6], connectivity [7], and routing [8].

The results in [7] focus on characterizing the fault tolerance of sensor networks by establishing the k-connectivity of a WSN. However, such characterization results in a poor lower bound of $k-1$ on the fault tolerance simply because the failure of some specific sets of k nodes partitions the network. Unfortunately, this is an extreme case behavior, and it fails to characterize the expected probability of network partitioning in practical deployments.

The results in [9, 10, 11, 12, 13] establish and explore the relationship between coverage and connectivity. The results in [11] and [12] show that in large sensor networks if the communication radius r_c is at least twice the coverage radius r_s, then complete coverage of a convex area implies connectivity among the working set of nodes. In [10], Bai *et al.* explore the relationship between coverage and connectivity if r_c/r_s is less than 2; they establish optimal coverage and connectivity in regular patterns including square grids and hexagonal lattice. However, maintaining connectivity in such scenarios requires deployment of additional sensors in periodic 'strips' across the region. The ratio r_c/r_s is weakened further in [9] to show that if $r_c/r_s = 1$ then, even if each node is highly unreliable, for large networks in a square region we can still maintain connectivity with coverage; however, as node failure probability increases, connectivity does not imply coverage. Ammari, *et al.*, extend these results in [13] with a focus on k-coverage: they show that if $r_c/r_s = 1$ in a k-covered homogeneous WSN, then the network fault tolerance is given by $4r_c(r_c + r_s)k/r_s^2 - 1$ as long as the entire neighborhood of any sensor does not fail at the same time. Another related result is [14] which shows that if a uniform random deployment of sensors in a WSN covers an entire area and $1 \le r_c/r_s \le 2$, then the probability of maintaining connectivity approaches 1 as r_c/r_s approaches 2.

A closely related work [15] explores the relationship among node density, transmission range, and k-connectivity in WSNs where the nodes are distributed uniformly at random. However, the results in [15] are applicable only for circular regions and they do not consider node failures in their analysis.

Our work differs from the works cited above in three aspects: (a) we focus exclusively on maintaining connectivity (and we are agnostic to coverage), (b) while the results in [9, 10, 11, 12] apply to specific deployment patterns or shape of a region, our results and methodology can be applied to any arbitrary region and any homogeneous deployment, and (c) our analysis is probabilistic insofar as node crashes are assumed to be independent random events and we assess the probability of maintaining connectivity despite such crashes; we focus on the probability of network connectivity in the average case instead of the worst case.

Apart from static analysis of coverage and connectivity, a significant body of work focuses on dynamic protocols to maintain coverage and/or connectivity by coordinating nodes to remain either active or asleep. Such schedules extend the lifetime of a network and reduce the overall power consumption. Examples of such protocols include AFECA [16], Naps [17], GAF [18], Span [19], ASCENT [20], PEAS [21], and partial clustering [22]. Our work, although related to the above, is orthogonal. Each of the above protocols behaves correctly only when the node distribution in the WSN is "adequately redundant"; we provide a quantitative measure of such a "adequate redundancy" by providing lower bounds on the probability of connectivity. Moreover, our results are obtained through mathematical analysis, instead of simulation and experimentation. Therefore, unlike simulation-based results, which are sensitive to the fidelity of the simulated runs to the real-world behavior, our results are robust and applicable to all homogeneous WSN deployments.

We assume that the node distribution in a WSN is homogeneous and the region is tiled by hexagons such that nodes in a given hexagon can communicate with the nodes in neighboring hexagons. Although a similar analysis could be done by tiling the region with squares (or any other polygon that tiles the plane), we chose a hexagonal tessellation for the following reason. Traditionally, the communication and sensing range of wireless sensors is approximated by a circle, and among the set of regular polygons that tile a plane, hexagons are the closest approximation to a circle.

Similar models have been used in [23], [24] and [25]. Liu *et al.* combine hexagonal tessellation of the plane with GAF [18] to derive energy-conserving deployment schemes for WSNs. Hexagonal tessellation is used in [24] for collision-free scheduling among nodes arranged in a lattice such that they can communicate with neighboring nodes in the lattice. Similarly, [25] uses a hexagonal tiling for collision-free scheduling in Mobile Ad-hoc NETworks.

The hexagonal lattice used in our analysis induces a hierarchical structure to the network, and this hierarchy can be used to decompose the connectivity property of a large network into connectivity properties of constituent smaller sub-networks of similar structure. This approach has been used earlier in analyzing the fault tolerance of interconnect networks [26, 27]. In [26], Chen *et al.* investigate the expected fault tolerance of an n-dimensional hypercube in the presence of node crashes. The metric for the expected fault tolerance is presented in terms of subcube-connectivity where it is assumed that smaller dimensional subcubes within the hypercube are connected, and this local connectivity property is shown to be sufficient for global connectivity. Similar methodology is adopted in [27] for mesh networks. Analogous to the subcube-connectivity in [26], the analysis in [27] employs the notion of submesh connectivity in which it is assumed that smaller meshes within the larger mesh are connected, and the global connectivity can be established from such local submesh connectivity. However, note that hypercube and mesh networks networks have extremely rigid topologies and so the results from [26, 27] do not carry over to WSN deployments. Our work follows a similar but different methodology wherein we construct incrementally larger polyhexes using the underlying hexagons to derive a recursive function that establishes a lower bound on network connectivity as a function of ρ and D.

3. System Model

We make the following simplifying assumptions: A WSN has a finite fixed set of n *nodes*. Each node has a communication radius R. A WSN *region* is assumed to be a closed topological disk on the Euclidean plane that is tiled by regular hexagons whose sides are of length l such that nodes located in a given hexagon can communicate reliably[2] with all the nodes in the same hexagon and adjacent hexagons. We assume that each hexagon contains at least D nodes. We assume that a node can fail only by *crashing* before the end of its intended lifetime. Node failures are independent and each node has a constant probability ρ of failing. A hexagon is said to be *empty* if it contains only faulty nodes.

We say that two non-faulty nodes p and p' are *connected* if either p and p' are in the same or neighboring hexagons, or there exists some sequence of non-faulty nodes $p_1, p_2, \ldots, p_i$ such that p (and p', respectively) and p_1 (and p_i, respectively) are in adjacent hexagons, and p_k and p_{k+1} are in adjacent hexagons, where $1 \leq k \leq i$. We say that a region is *connected* if every pair of non-faulty nodes p and p' in the region are connected.

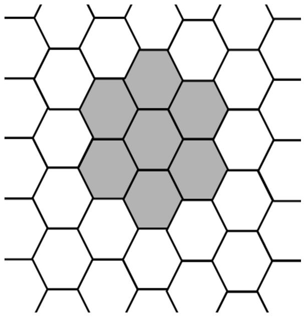
3.1. Level-z Polyhexes

For the analysis of WSNs in an arbitrary region, we use the notion of higher level tilings by grouping sets of contiguous hexagons into 'super tiles' such that some specific properties (like the ability to tile the Euclidean plane) are preserved. Such 'super tiles' are called level-z polyhexes. Different values of z specify different level-z polyhexes as follows.

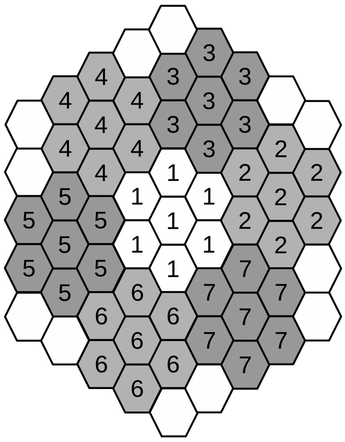
Definition 1. *A level-z polyhex for $z \in \mathbb{N}$ is defined as follows:*

- *A level-1 polyhex is a regular hexagon, and each side of a level-1 polyhex is the hexagon itself.*
- *A level-z polyhex for $z \geq 2$ is a union of non-overlapping hexagons (a* trans-polyhex*) which satisfies the following properties:*
    1. *A level-z polyhex is a union of seven non-overlapping level-$(z-1)$ polyhexes. Among the seven polyhexes, one polyhex is denoted* internal *and the remaining six are called* external *polyhexes.*
    2. *The internal polyhex is adjacent to all the external polyhexes, and each external polyhex is adjacent to the internal polyhex and to two other external polyhexes.*

[2] We assume that collision resolution techniques are always successful in ensuring reliable communication.



(a) The gray tiles form a level-2 polyhex.



(b) A level-3 polyhex formed by 7 level-2 polyhexes.

Figure 1: Examples of Polyhexes

3. *A level-z polyhex has six* sides*, where each side is the union of two sides of a level-*$(z-1)$ *polyhex and one side of another level-*$(z-1)$ *polyhex, such that the union is a closed topological disk*[3].

The Appendix contains all the technical definitions and framework for a formal specification of a level-z polyhex. We assert that for any z, a level-z polyhex exists and tiles the plane. The formal proof for this assertion is available in the appendix.

Recall that a side of a polyhex is a union of hexagons. For convenience, we denote the number of hexagons in a side of a polyhex as its *length*; the length of a level-z polyhex is given by

$$length(z) = 1 + \sum_{i=0}^{z-2} 3^i$$

The formal proof for the above claim is also in the Appendix.

4. Level-z Polyhexes and Connectivity

The analysis in Section 5 is based on the notion of *level-z connectedness* that is introduced here. First, we introduce the concepts of *connected level-z*

[3] A closed topological disk is the image of a closed circular disk under a homeomorphism. Roughly speaking, a homeomorphism is a continuous stretching and bending of the object into a new shape (tearing or 'cutting holes' into the object is not allowed). Thus, any two dimensional shape that has a closed boundary, finite area, and no 'holes' is a closed topological disk. This includes squares, circles, ellipses, hexagons, and polyhexes.

polyhexes and *level-z connectedness* in a WSN region, and then, we show how level-z connectedness implies that all non-faulty nodes in a WSN region are connected. We use this result and the definition of level-z connectedness to derive a lower bound on the probability of network connectivity in Section 5.

Connected level-z polyhex. Intuitively, we say that a level-z polyhex is *connected* if the network of nodes in the level-z polyhex is not partitioned. Formally, a level-z polyhex T_{zi} is said to be *connected* if, given the set Λ of all hexagons in T_{zi} that contain at least one non-faulty node, for every pair of hexagons p and q from Λ, there exists some (possibly empty) sequence of hexagons $t_1, t_2, \ldots, t_j$ such that $\{t_1, t_2, \ldots, t_j\} \subseteq \Lambda$, and t_1 is a neighbor of p, every t_i is a neighbor of t_{i+1}, and t_j is a neighbor of q.

Note that if a level-z polyhex is connected, then all the non-faulty nodes in the level-z polyhex are connected as well. We are now ready to define the notion of *level-z connectedness* in a WSN region.

Level-z connectedness. A WSN region $\mathcal{W}$ is said to be *level-z connected* if there exists some partitioning of $\mathcal{W}$ into disjoint level-z polyhexes such that (1) each such level-z polyhex is connected, and for every pair of such level-z polyhexes T_{zp} and T_{zq}, there exists some (possibly empty) sequence of connected level-z polyhexes $T_{z1}, T_{z2}, \ldots, T_{zj}$ (from the partitioning of $\mathcal{W}$) such that T_{z1} is a neighbor of T_{zp}, every T_{zi} is a neighbor of $T_{z(i+1)}$, and T_{zj} is a neighbor of T_{zq}; (2) each side of T_{zi} has more than $\lceil \frac{length(z)}{2} + 2^{z-4} \rceil$ non-empty hexagons.

The motivation for level-z connectedness is as follows. For a contiguous WSN region $\mathcal{W}$ to be connected, it is sufficient if the following two conditions are satisfied: (1) there exists a partitioning of $\mathcal{W}$ into disjoint level-z polyhexes (for some z) such that each such level-z polyhex is connected, and (2) between two adjacent such level-z polyhexes there exist sufficient non-empty hexagons on the sides of the polyhexes such that two adjacent non-empty hexagons, one from each polyhex, form a "bridge" between the two polyhexes, thus ensuring that the entire region $\mathcal{W}$ is connected.

While part (1) of the definition of level-z connectedness is straightforward, the specification of part (2) of the definition merits explanation. Between two adjacent level-z polyhexes, at first glance, it may seem that $\lceil \frac{length(z)}{2} \rceil$ non-faulty hexagons on each side is enough to form the aforementioned "bridge". However, because of the non-convex nature of the border of a level-z polyhex, a greater number of hexagons are necessary to ensure such a bridge exists. For example, consider a level-3 polyhex. Note that $\lceil \frac{length(3)}{2} \rceil = 3$. However, we can see on Figure 2, we need at least 4 non-empty hexagons on each side of a level-3 polyhex to ensure that it is connected to an adjacent level-3 polyhex. We will show next how the number of non-empty hexagons on each side sufficient to ensure connectivity is more than $\lceil \frac{length(z)}{2} + 2^{z-4} \rceil$.

Lemma 1. *Let T_1 and T_2 be two adjacent level-z polyhexes, such that the hexagons from side S_1 of T_1 are adjacent to the hexagons from side S_2 from T_2. If S_1 and S_2 each contain at least $\lceil \frac{length(z)}{2} + 2^{z-4} \rceil$ non-empty hexagons,*

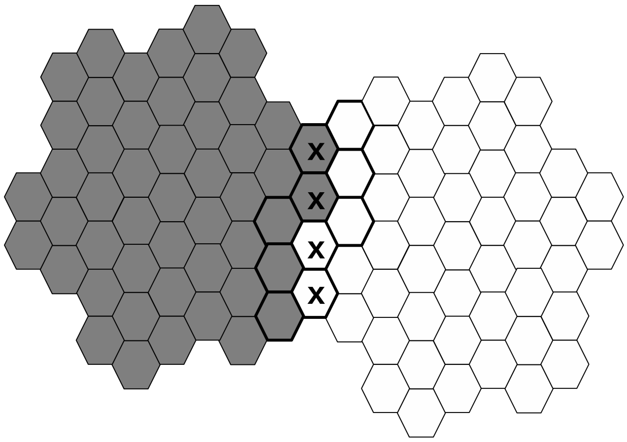


Figure 2: This figure illustrates how a simple majority of non-empty hexagons along the sides of two level-3 polyhexes can still partition the network. Here hexagons with thick outline denote the hexagons on the sides of the level-3 polyhexes along which the polyhexes are connected to each other. The hexagons with an '**x**' in them denote empty hexagons. Note that *length*(3) = 5, and despite at least 3 non-empty hexagons on each side of the two level-3 polyhexes, the network consisting of two level-3 polyhexes is still partitioned.

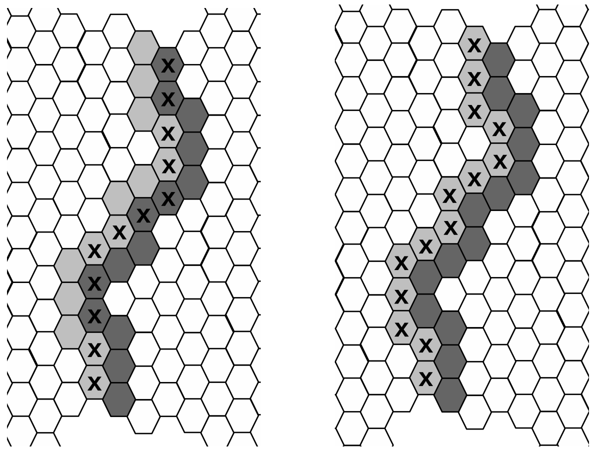
then there exists a pair of hexagons $h_1 \in S_1$ *and* $h_2 \in S_2$*, such that* h_1 *and* h_2 *are adjacent and non-empty.*

Proof. For $z = 1$ and $z = 2$, we see that $\lceil \frac{length(z)}{2} + 2^{z-4} \rceil = length(z)$. Hence, the lemma is true for $z \leq 2$. The remainder of this proof considers the case $z > 2$.

In order to determine the number of non-faulty hexagons needed on S_1 and S_2 we first calculate the smallest possible number of empty hexagons on S_1 and S_2 combined, which causes T_1 and T_2 to be disconnected. Let this value be denoted as $dist(z)$. In other words, $dist(z)$ is the length of the shortest possible path consisting of adjacent empty hexagons on either S_1 or S_2 which disconnect the two level-z polyhexes (for example, Figure 3(a) illustrates shortest possible path for $z = 4$). Given $dist(z)$ we can determine how many non-empty hexagons are required on both sides so that if T_1 is connected and T_2 is connected, then the union of T_1 and T_2 is also connected. The total number of hexagons on both S_1 and S_2 together is $2 \cdot length(z)$; therefore, the minimum number of non-empty hexagons needed on both sides is strictly greater than $2 \cdot length(z) - dist(z)$.

It is easy to see that $dist(z)$ is upper bounded by $length(z)$ because if all the hexagons on a side (say) S_1 are empty, then T_1 and T_2 are no longer connected. However, due to the non-convex nature of the sides of a level-k polyhex, it is possible to have a shorter path by choosing hexagons from both polyhexes to be empty. Consider the case where $z = 3$; that is, consider level-3 polyhexes. As seen in Figure 2, we can verify that $length(3) = 5$ whereas $dist(3) = 4$; that is $length(3) - dist(3) = 1$.

Note that for any $z > 3$, all level-z polyhexes can be constructed as a union



(a) The shortest path consisting of 12 adjacent empty hexagons on either side of two adjacent polyhexes that partitions the region.

(b) The path consisting of 14 empty hexagons from a single side that partitions the region.

Figure 3: Empty hexagons that partition adjacent polyhexes. In the above two figures, the empty hexagons are marked with an '**x**' in the center.

of constituent (non-overlapping) level-3 polyhexes. Consequently, for any two adjacent level-z polyhexes T_1 and T_2 and the corresponding sides S_1 and S_2, if the empty hexagons in $S_1 \cup S_2$ partition the region formed by the union of T_1 and T_2, then we know that every such empty hexagon must belong to some constituent level-3 polyhex. However, since we know from Figure 2 that $length(3) - dist(3) = 1$, we conclude that $length(z) - dist(z)$ is given by the number of level-3 polyhexes that contribute hexagons to a side of the level-z polyhex.

As an example, Figures 3(a) and 3(b) show the difference between the length of a side and the actual shortest distance on the sides of two level-4 polyhexes. Each side of a level-4 polyhex consists of two level-3 polyhexes. Hence, $length(z) - dist(z) = 2$. We extend this to higher level-z polyhexes as follows.

By the definition of a level-z polyhex, each of its sides consists of two distinct level-$(z-1)$ polyhexes. Therefore, a side of a level-z polyhex contains hexagons from 2^{z-3} level-3 polyhexes, and so $length(z) - dist(z) = 2^{z-3}$. By the definition of $dist(z)$, if there are more than $length(z) - 2^{z-3}$ non-empty hexagons on both sides S_1 and S_2, then the two polyhexes are guaranteed to be connected. Therefore, if each side contains more than $\lceil \frac{length(z)}{2} + 2^{z-4} \rceil$ hexagons, then T_1 and T_2 are guaranteed to be connected. $\square$

We are now ready to prove the following theorem:

Theorem 2. *Given a WSN region $\mathcal{W}$, if $\mathcal{W}$ is level-z connected for some z, then all non-faulty nodes in $\mathcal{W}$ are connected.*

Proof. Suppose that the region $\mathcal{W}$ is level-z connected for some z. It follows that there exists some partitioning Λ of $\mathcal{W}$ into disjoint level-z polyhexes such that each such level-z polyhex is connected, and for every pair of such level-z polyhexes T_{zp} and T_{zq}, there exists some (possibly empty) sequence of connected level-z polyhexes $T_{z1}, T_{z2}, \ldots, T_{zj}$ (from the partitioning of $\mathcal{W}$) such that T_{z1} is a neighbor of T_{zp}, every T_{zi} is a neighbor of $T_{z(i+1)}$, and T_{zj} is a neighbor of T_{zq}. Additionally, each side of T_{zi} has more than $\lceil \frac{length(z)}{2} + 2^{z-4} \rceil$ non-empty hexagons.

To prove the theorem, it is sufficient to show that for any two non-faulty nodes in $\mathcal{W}$ in hexagons p and q, respectively, the hexagons p and q are connected.

Let hexagon p lie in a level-z polyhex $T_{zp} \in \Lambda$, and let q lie in a level-z polyhex $T_{zq} \in \Lambda$. Note that since Λ is a partitioning of $\mathcal{W}$, either $T_{zp} = T_{zq}$ or T_{zp} and T_{zq} are disjoint. If $T_{zp} = T_{zq}$, then since T_{zp} is connected, it follows that p and q are connected. Hence, all non-faulty nodes in p are connected with all non-faulty nodes in q. Thus, the theorem is satisfied.

If T_{zp} and T_{zq} are disjoint, then it follows from the definition of level-z connectedness that there exists some sequence of connected level-z polyhex $T_{z1}, T_{z2}, \ldots, T_{zj}$ such that T_{z1} is a neighbor of T_{zp}, every T_{zi} is a neighbor of $T_{z(i+1)}$, and T_{zj} is a neighbor of T_{zq}.

Consider any two neighboring level-z polyhexes $(T_{zm}, T_{zn}) \in \Lambda \cdot \Lambda$. Since each side of T_{zi} has more than $\lceil \frac{length(z)}{2} + 2^{z-4} \rceil$ non-empty hexagons, we know from Lemma 1 that there exist adjacent non-empty hexagons h_m and h_n such that h_m is from T_{zm} and h_n is from T_{zn}. Therefore, h_m and h_n form a "bridge" between T_{zm} and T_{zn} allowing nodes in T_{zm} to communicate with nodes in T_{zn}. Since T_{zm} and T_{zn} are connected level-z polyhexes, it follows that nodes within T_{zm} and T_{zn} are connected as well. Additionally, we have established that there exist at least two hexagons, one in T_{zm} and one in T_{zn} that are connected. It follows that nodes in T_{zm} and T_{zn} are connected with each other as well.

Thus, it follows that T_{zp} and T_{z1} are connected, every T_{zi} is connected with $T_{z(i+1)}$, and T_{zj} is connected with T_{zq}. From the transitivity of connectedness, it follows that T_{zp} is connected with T_{zq}. That is, all non-faulty nodes in hexagon p are connected with all non-faulty nodes in q. Since p and q are arbitrary hexagons in $\mathcal{W}$, it follows that all the nodes in $\mathcal{W}$ are connected. □

5. On Fault Tolerance of WSN Regions

We are now ready to derive a lower bound on the connectivity probability of an arbitrarily-shaped WSN region. Let $\mathcal{W}$ be a WSN region with node density of D nodes per hexagon such that the region tiled by a patch of x level-z polyhexes that constitute a set Λ. Let each node in the region fail independently with probability ρ. Let $Conn_{\mathcal{W}}$ denote the event that all the non-faulty nodes in the region $\mathcal{W}$ are connected. Let $Conn_{(T,z,side)}$ denote the event that a level-z polyhex T is connected and each side of T has more than $\lceil \frac{length(z)}{2} + 2^{z-4} \rceil$ non-empty hexagons.

We know that if $\mathcal{W}$ is level-z connected, then all the non-faulty nodes in $\mathcal{W}$ are connected. Also, $\mathcal{W}$ is level-z connected if: $\forall T \in \Lambda :: Conn_{(T,z,side)}$. Therefore, the probability that $\mathcal{W}$ is connected is bounded by: $Pr\left[Conn_{\mathcal{W}}\right] \geq (Pr\left[Conn_{(T,z,side)}\right])^x$. Thus, in order to find a lower bound on $Pr\left[Conn_{\mathcal{W}}\right]$, we compute a lower bound on $(Pr\left[Conn_{(T,z,side)}\right])^x$.

For the remainder of this paper, the number of hexagons in a level-z polyhex is denoted $size(z)$. Since each level-z polyhex (for $z > 1$) consists of seven level-$(z-1)$ polyhexes, we can easily see that $size(z) = 7^{z-1}$.

Lemma 3. *In a level-z polyhex T with node density of D nodes per hexagon, suppose each node fails independently with a probability ρ. Then the probability that T is connected and each side of T has more than $\lceil \frac{length(z)}{2} + 2^{z-4} \rceil$ non-empty hexagons is given by*

$$Pr\left[Conn_{(T,z,side)}\right] = \sum_{i=0}^{size(z)} N_{z,i}(1-\rho^D)^{size(z)-i}\rho^{D\cdot i}$$

where $N_{z,i}$ is the number of ways by which we can have i empty hexagons and $size(z)-i$ non-empty hexagons in a level-z polyhex such that the level-z polyhex is connected and each side of the level-z polyhex has more than $\lceil \frac{length(z)}{2} + 2^{z-4} \rceil$ non-empty hexagons.

Proof. Fix i hexagons in T to be empty such that T is connected and each side of T has more than $\lceil \frac{length(z)}{2} + 2^{z-4} \rceil$ non-empty hexagons. Since nodes fail independently with probability ρ, and there are D nodes per hexagon, the probability that a hexagon is empty is ρ^D. Therefore, the probability that exactly i hexagons are empty in T is given by $(1-\rho^D)^{size(z)-i}\rho^{D\cdot i}$. By assumption, there are $N_{z,i}$ ways to fix i hexagons to be empty. Therefore, the probability that T is connected and each side of T has more than $\lceil \frac{length(z)}{2} + 2^{z-4} \rceil$ non-empty hexagons despite i empty hexagons is given by $N_{z,i}(1-\rho^D)^{size(z)-i}\rho^{D\cdot i}$. However, note that we can set i (the number of empty hexagons) to be anything from 0 to $size(z)$. Therefore, $Pr\left[Conn_{(T,z,side)}\right]$ is given by $\sum_{i=0}^{size(z)} N_{z,i}(1-\rho^D)^{size(z)-i}\rho^{D\cdot i}$. □

Given the probability of $Conn_{(T,z,side)}$, we can now establish a lower bound for the probability that the region $\mathcal{W}$ is connected.

Theorem 4. *Suppose each node in a WSN region $\mathcal{W}$ fails independently with probability ρ, $\mathcal{W}$ has a node density of D nodes per hexagon and is tiled by a patch of x level-z polyhexes. Then the probability that all non-faulty nodes in $\mathcal{W}$ are connected is at least $(Pr\left[Conn_{(T,z,side)}\right])^x$*

Proof. There are x level-z polyhexes in $\mathcal{W}$. Note that if $\mathcal{W}$ is level-z connected, then all non-faulty nodes in $\mathcal{W}$ are connected. However, observe that $\mathcal{W}$ is level-z connected if each such level-z polyhex is connected and each side of each such level-z polyhex has more than $\lceil \frac{length(z)}{2} + 2^{z-4} \rceil$ non-empty hexagons. Recall from Lemma 3 that the probability of such an event for each polyhex is given by $Pr\left[Conn_{(T,z,side)}\right]$. Since there are x such level-z polyhex, and failure probability of nodes (and hence disjoint level-z polyhexes) is independent, it follows that the probability of $\mathcal{W}$ being connected is at least $(Pr\left[Conn_{(T,z,side)}\right])^x$. □

Note that the lower bound we have established depends on the function $N_{z,i}$ defined in Lemma 3. Unfortunately, to the best of our knowledge, there is no known algorithm that computes $N_{z,i}$ in a reasonable amount of time. Since this is a potentially infeasible approach for large WSNs with millions of nodes, we provide an alternate lower bound for $Pr\left[Conn_{(T,z,side)}\right]$ that is easier to calculate. In the following lemma, recall that $size(z)$ denotes the number of hexagons in a level-z polyhex and $size(z) = 7^{z-1}$.

Lemma 5. *The value of* $Pr\left[Conn_{(T,z,side)}\right]$ *from Lemma 3 is bounded below by* $Pr\left[Conn_{(T,z,side)}\right] \geq (Pr\left[Conn_{(T,z-1,side)}\right])^7 + (Pr\left[Conn_{(T,z-1,side)}\right])^6 \cdot \rho^{D\cdot size(z-1)}$*, where* $Pr\left[Conn_{(T,1,side)}\right] = 1 - \rho^D$*.*

Proof. Recall that a level-z polyhex consists of seven level-$(z-1)$ polyhexes with one internal level-$(z-1)$ polyhex and six outer level-$(z-1)$ polyhexes. Observe that a level-z polyhex satisfies $Conn_{(T,z,side)}$ if either (a) all the seven level-$(z-1)$ polyhexes satisfy $Conn_{(T,z-1,side)}$, or (b) the internal level-$(z-1)$ polyhex is empty and the six outer level-$(z-1)$ polyhexes satisfy $Conn_{(T,z-1,side)}$. From Lemma 3 we know that the probability of a level-$(z-1)$ polyhex satisfying $Conn_{(T,z-1,side)}$ is given by $Pr\left[Conn_{(T,z-1,side)}\right]$ and the probability of a level-$(z-1)$ polyhex being empty is $\rho^{D\cdot size(z-1)}$. For a level-1 polyhex (which is a regular hexagon tile), the probability that the hexagon is not empty is $1-\rho^D$. Therefore, the probability that cases (a) or (b) is satisfied for $z > 1$ is given by $(Pr\left[Conn_{(T,z-1,side)}\right])^7+(Pr\left[Conn_{(T,z-1,side)}\right])^6\cdot\rho^{D\cdot size(z-1)}$. Therefore, $Pr\left[Conn_{(T,z,side)}\right] \geq (Pr\left[Conn_{(T,z-1,side)}\right])^7 + (Pr\left[Conn_{(T,z-1,side)}\right])^6 \cdot \rho^{D\cdot size(z-1)}$ where $Pr\left[Conn_{(T,1,side)}\right] = 1 - \rho^D$. □

Analyzing the connectivity probability for WSN regions that are level-z connected where z is large, can be simplified by invoking Lemma 5, and reducing the complexity of the computation to smaller values of z for which $Pr\left[Conn_{(T,z,side)}\right]$ can be computed (by brute force) fairly quickly.

5.1. WSN region as a level-z polyhex

A special case considered is where the WSN region $\mathcal{W}$ is a single level-z polyhex[4]. For $\mathcal{W}$ to be connected, it is sufficient if the level-z polyhex is connected; it is not necessary for the sides of the level-z polyhex to contain more than $\lceil \frac{length(z)}{2} + 2^{z-4} \rceil$ non-empty hexagons. Based on this argument, we have the following theorem.

[4]Note that a level-z polyhex provides an adequate approximation for a large circular region. Although this is a special case relative to the analysis in the previous section, circular WSN regions are not an uncommon deployment in practice.

Theorem 6. *Let each node in a WSN region $\mathcal{W}$, with node density D nodes per hexagon, fail independently with probability ρ, and let $\mathcal{W}$ be tiled by a single level-z polyhex. Then the lower bound on the probability that all non-faulty nodes in $\mathcal{W}$ are connected is given by $Pr\left[Conn_{\mathcal{W}}\right]$ where $Pr\left[Conn_{\mathcal{W}}\right] \geq Pr\left[Conn_{(T,z)}\right]$, and*

$$Pr\left[Conn_{(T,z)}\right] \geq \begin{cases} \rho^{D\cdot size(z)} \\ +7(Pr\left[Conn_{(T,z-1)}\right])(\rho^{D\cdot size(z-1)})^6 \\ +12(Pr\left[Conn_{(T,z-1,side)}\right])^2(\rho^{D\cdot size(z-1)})^5 \\ +15(Pr\left[Conn_{(T,z-1,side)}\right])^3(\rho^{D\cdot size(z-1)})^4 \\ +15(Pr\left[Conn_{(T,z-1,side)}\right])^4(\rho^{D\cdot size(z-1)})^3 \\ +12(Pr\left[Conn_{(T,z-1,side)}\right])^5(\rho^{D\cdot size(z-1)})^2 \\ +7(Pr\left[Conn_{(T,z-1,side)}\right])^6(\rho^{D\cdot size(z-1)}) \\ +(Pr\left[Conn_{(T,z-1,side)}\right])^7 \end{cases}$$

where $Pr\left[Conn_{(T,1)}\right] = 1$ and $Pr\left[Conn_{(T,1,side)}\right] = 1 - \rho^D$.

Proof. Let each node in a WSN region $\mathcal{W}$, with node density of D nodes per hexagon, fail independently with probability ρ, and let $\mathcal{W}$ be tiled by a single level-z polyhex. Recall that the level-z polyhex tiling $\mathcal{W}$ consists of seven level-$(z-1)$ polyhexes. $\mathcal{W}$ will be connected if the any of the following conditions are true:

1. **All the nodes crash**: If all nodes crash, then the region is connected by default. The probability of this event is $\rho^{D\cdot size(z)}$.
2. **One level-$(z-1)$ polyhex is connected, and all other hexagons in the remaining 6 are empty**: The probability that a level-$(z-1)$ polyhex is connected is given by $Pr\left[Conn_{(T,z-1)}\right]$ and the probability that all the nodes in a level-$(z-1)$ polyhex are crashed is $\rho^{D\cdot size(z-1)}$. Since there are 7 ways this could happen, the probability of this event is $7(Pr\left[Conn_{(T,z-1)}\right])(\rho^{D\cdot size(z-1)})^6$.
3. **Two adjacent level-$(z-1)$ polyhexes satisfy $Conn_{(T,z-1,side)}$, and all other hexagons in the remaining 5 are empty**: Note that if two adjacent level-$(z-1)$ polyhexes satisfy $Conn_{(T,z-1,side)}$, then all the non-faulty nodes in the two level-$(z-1)$ polyhexes are connected. Since there are 12 ways this could happen, the probability of this event is given by $12(Pr\left[Conn_{(T,z-1,side)}\right])^2 \cdot (\rho^{D\cdot size(z-1)})^5$.
4. **Three adjacent level-$(z-1)$ polyhexes satisfy $Conn_{(T,z-1,side)}$, and all other hexagons in the remaining 4 are empty**: This case is similar to the previous case except that there are 15 ways to choose three adjacent level-$(z-1)$ polyhexes from the level-z polyhex. Therefore, the probability of this event is $15(Pr\left[Conn_{(T,z-1,side)}\right])^3 \cdot(\rho^{D\cdot size(z-1)})^4$.
5. **Four adjacent level-$(z-1)$ polyhexes satisfy $Conn_{(T,z-1,side)}$, and all other hexagons in the remaining 3 are empty**: This case is

similar to the previous one. Therefore, the probability of this event is $15(Pr\left[Conn_{(T,z-1,side)}\right])^4 \cdot (\rho^{D\cdot size(z-1)})^3$.

6. **Five adjacent level-$(z-1)$ polyhexes satisfy $Conn_{(T,z-1,side)}$, and all other hexagons in the remaining 2 are empty**: This case is similar to the earlier cases: there are 12 ways to choose five adjacent level-$(z-1)$ polyhexes from the level-z polyhex. Therefore, the probability of this event is
$12(Pr\left[Conn_{(T,z-1,side)}\right])^5 \cdot (\rho^{D\cdot size(z-1)})^2$.
7. **Six adjacent level-$(z-1)$ polyhexes satisfy $Conn_{(T,z-1,side)}$, and all other hexagons in the remaining level-$(z-1)$ polyhex are empty**: This case is similar to the earlier cases: there are 7 ways to choose six adjacent level-$(z-1)$ polyhexes from the level-z polyhex. Therefore, the probability of this event is by $7(Pr\left[Conn_{(T,z-1,side)}\right])^6 \cdot (\rho^{D\cdot size(z-1)})$.
8. **All the level-$(z-1)$ polyhexes satisfy $Conn_{(T,z-1,side)}$**: The probability of this event is by
$(Pr\left[Conn_{(T,z-1,side)}\right])^7$.

For the base case, let $z = 1$. Then, $Pr\left[Conn_{(T,z)}\right]$ corresponds to a single hexagon. By definition the probability of a hexagon being connected to itself is 1; that is, $Pr\left[Conn_{(T,1)}\right] = 1$. Note that all the sides of the hexagon consist of the hexagon itself. Therefore, $Pr\left[Conn_{(T,1,side)}\right]$ corresponds to the probability that the hexagon is non-empty, which is $1 - \rho^D$. Combining all of the above with the base case, the theorem is proved. □

In practice, exhaustive enumeration can be used to compute the probability for a base case greater than 1 for a tighter lower bound on fault tolerance of such WSNs.

6. Discussion

Hexagons vs. Other Tilings. In our investigations, we tiled the WSN region using hexagons. Note that we may tile the region using other polygons, such as a square or triangle, that admit tessellations. However, choosing a hexagonal tessellation offers the following advantage. A regular hexagon and a level-2 polyhex are fairly good approximations of circular regions. Since our analysis assumes that nodes in a given tile can communicate with nodes in all the neighboring tiles, and given that communication range is assumed to circular, we see that a regular hexagon models such behavior better than other polygons such a square or a triangle.

Choosing the size of the hexagon. For the results from the previous section to be of practical use, it is important that we choose the size of the hexagons in our system model carefully. On the one hand, choosing very large hexagons could violate the system model assumption that nodes can communicate with nodes in neighboring hexagons, and on the other hand, choosing small hexagons could

z	i	$N_{z,i}$	z	i	$N_{z,i}$
$k > 2$	1	$size(k) = 7^{k-1}$	3	9	382346951
3	2	1116	4	2	58653
3	3	15802	4	3	6666849
3	4	156868	4	4	566671431
3	5	1166394	4	5	38418864159
3	6	6771841	5	2	2881200
3	7	31574141	5	3	2303999486
3	8	120576864	5	4	1381247486690

Table 1: Computed Values of $N_{z,i}$

Node density D	No. of Nodes	Node failure prob. ρ	No. of Nodes	Node failure prob. ρ
	$z = 3$ (level-3 polyhex)		$z = 5$ (level-5 polyhex)	
3	137	31%	7203	17%
5	245	49%	12005	34%
10	490	70%	24010	58%
	$z = 4$ (level-4 polyhex)		$z = 7$ (level-7 polyhex)	
3	1029	22%	352947	13%
5	1715	41%	588245	29%
10	3430	64%	1176490	53%

Table 2: Various values for node failure probability ρ, node density D, and level-z polyhex that yield network connectivity probability exceeding 95%

result in poor lower bounds and thus result in over-engineered WSNs that incur high costs but with incommensurate benefits.

If we make no assumptions about the locations of nodes within hexagons, then the length l of the sides of a hexagon must be at most $R/\sqrt{13}$ to ensure connectivity between non-faulty nodes in neighboring hexagons[5]. However, if the nodes are "evenly" placed within each hexagon, then l can be as large as $R/2$ while still ensuring connectivity between neighboring hexagons[6]. In both cases, the requirement is that the distance between two non-faulty nodes in neighboring hexagons is at most R.

Arbitrarily Shaped Regions. Note that our results can be applied to arbitrarily shaped regions as follows. Given any arbitrary region, first tile the region with hexagons of appropriate size (as discussed earlier). This will determine the node

[5]The reader can verify that, given two adjacent regular hexagons whose sides are of length l, the largest distance between any two points, one in each hexagon, does not exceed $l\sqrt{13}$.

[6]The bound $l \leq R/2$ is a consequence of the following observation: in a plane tiled by regular hexagons whose sides are of length l, any two points that are $2l$ distance apart are either in the same hexagon or adjacent hexagons.

density D of the region. Next, determine the highest z, such that a patch of x level-z polyhexes tiles that region. Now, we can apply Theorem 4 to determine a lower bound on the probability of connectivity.

Computing $N_{z,i}$ from Lemma 3. The function $N_{z,i}$ does not have a closed-form solution. It needs to be computed through exhaustive enumeration. We computed $N_{z,i}$ for some useful values of z and i and included them in Table 1. Using these values, we applied Theorem 4 and Lemma 5 to sensor networks of different sizes, node densities, and node failure probabilities. The results are presented in Table 2. Next, we demonstrate how to interpret and understand the entries in these tables through an illustrative example.

Practicality. Our results can be utilized in the following two practical scenarios. (1) Given an existing WSN with known node failure probability, node density, and area of coverage, we can estimate the probability of connectivity of the entire network. First, we decide on the size of a hexagon as discussed previously, and then we consider level-z polyhexes that cover the region. Next, we apply Theorem 4 and Lemma 5 (or Theorem 6, if applicable) to compute the probability of connectivity of the network for the given values of ρ, D and z, and the precomputed values of $N_{z,i}$ in Table 1.

(2) The results in this paper can be used to design a network with a specified probability of connectivity. In this case, we decide on a hexagon size that best suits the purposes of the sensor network and determine the level of the polyhex(es) needed to cover the desired area. As an example, consider a 200 sq. km region (approximately circular, so that there are no 'bottle neck' regions) that needs to be covered by a sensor network with a 95% connectivity probability. Let the communication radius of each sensor be 50 meters. The average-case value of the length l of the side of the hexagon is 25 meters, and the 200 sq. km region is tiled by a single level-7 polyhex. From Table 2, we see that if the network consists of 3 nodes per hexagon, then the region will require about 352947 nodes with a failure probability of 13% (87% reliability). However, if the node redundancy is increased to 5 nodes per hexagon, then the region will require about 588245 nodes with a failure probability of 29% (71% reliability). If the node density is increased further to 10 nodes per hexagon, then the region will require about 1176490 nodes with a failure probability of 53% (47% reliability).

On the lower bounds. An important observation is that these values for node reliability are lower bounds, but are definitely not tight bounds. This is largely because in order to obtain tighter lower bounds, we need to compute the probability of network connectivity from Theorem 4. However, this requires us to compute the values for $N_{z,i}$ for all values of i ranging from 1 to z, which is expensive for z exceeding 3. Consequently, we are forced to use the recursive functions in Lemma 5 and Theorem 6 for computing the network connectivity for larger networks. This reduces the accuracy of the lower bound significantly. A side effect of this error is that in Table 2, we see that for a given D, ρ decreases as z increases. If we were to invest the time and computing resources to compute $N_{z,i}$ for higher values of z (5, 7, and greater), then the computed values for ρ in Table 2 would be significantly larger.

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Appendix A. Higher level tilings: polyhexes

For the analysis of WSNs in an arbitrary region, we use the notion of higher level tilings by grouping sets of contiguous hexagons into 'super tiles' such that some specific properties (like the ability to tile the Euclidean plane) are preserved. Such 'super tiles' are called level-z polyhexes. Different values of z specify different level-z polyhexes. In this section we define a level-z polyhex, prove its existence and specify its properties.

Appendix A.1. Definitions

These definitions have been borrowed from [28]:

> A *tiling* of the Euclidean plane is a countable family of closed sets called tiles, such that the union of the sets is the entire plane and such that the interiors of the sets are pairwise disjoint. We are concerned only with *monohedral* tilings — tilings in which every tile is congruent[7] to a single fixed tile called the *prototile*. We say that the prototile *admits* the tiling.

[7] Recall that two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry, *i.e.*, a combination of translations, rotations and reflections.

> [...]
>
> A *patch* is a finite collection of non-overlapping tiles such that their union is a closed topological disk[8]. A *translational patch* is a patch such that the tiling consists entirely of a lattice of translations[9] of that patch.

We now define a translational patch of regular hexagons called a *trans-polyhex* and a special family of trans-polyhexes called *level-z polyhexes* for $z \in \mathbb{N}$.

Definition 2. *A trans-polyhex is the union of a set S of non-overlapping hexagons, which forms a translational patch.*

A trans-polyhex, by virtue of being a translational patch, tiles the Euclidean plane and satisfies the *translation criterion* as defined in [28]:

> A closed topological disk satisfies the *translation criterion* if you can divide up the boundary into six segments labeled clockwise A, B, C, D, E, and F such that each of the three pairs A-D, B-E, and C-F are translations of each other (both edges in one of these pairs may be empty).
>
> [...]
>
> A prototile satisfying the translation criterion admits a lattice-translation tiling simply by translating copies so that the edges in each pair line up. If one of the pairs is empty, then the tiling forms a rectangular lattice, otherwise it forms a hexagonal lattice.

The theorem from [28] associated with the translation criterion states:

Theorem 7. *A prototile that is a closed topological disk admits a tiling of the plane by a lattice of translations, if and only if, the prototile satisfies the translation criterion.*

Based on the translation criterion, we define sides of a trans-polyhex as follows.

Definition 3. *A trans-polyhex H has six* sides *S_A–S_F where each side S_i is the set of hexagons from H which contribute at least one line segment to the boundary segment i in the translation criterion.*

Note that the above definition need not uniquely determine the sides of a trans-polyhex. We resolve this ambiguity in the definition of a level-z polyhex in Definition 5. In order to define a level-z polyhex, we have to first define what it means for two trans-polyhexes to be *adjacent*.

[8] A closed topological disk is the image of a closed circular disk under a homeomorphism. Roughly speaking, a homeomorphism is a continuous stretching and bending of the object into a new shape (tearing the object or 'cutting holes' into the object is not allowed). Thus, any two dimensional shape that has a closed boundary, finite area, and no 'holes' is a closed topological disk. This includes squares, circles, ellipses, hexagons, and polyhexes.

[9] Recall that a translation is moving every point a constant distance in a specified direction.

Definition 4. *Two trans-polyhexes T_1 and T_2 are adjacent if there exist a side S_1 of T_1 and a side S_2 of T_2 such that each hexagon in S_1 is adjacent to some hexagon in S_2 or vice versa, and the region formed by the union of T_1 and T_2 is a closed topological disc.*

We are now ready to define a level-z polyhex.

Definition 5. *A level-z polyhex for $z \in \mathbb{N}$ is defined as follows:*

- *A level-1 polyhex is a regular hexagon, and each side of a level-1 polyhex is the hexagon itself.*
- *A level-z polyhex for $z \geq 2$ is a trans-polyhex which satisfies the following properties:*
  1. *A level-z polyhex is a union of seven non-overlapping level-$(z-1)$ polyhexes. Among the seven polyhexes, one polyhex is denoted* internal *and the remaining six are called* external *polyhexes.*
  2. *The internal polyhex is adjacent to all the external polyhexes, and each external polyhex is adjacent to the internal polyhex and to two other external polyhexes.*
  3. *A level-z polyhex has six* sides*, where each side is the union of two sides of a level-$(z-1)$ polyhex and one side of another level-$(z-1)$ polyhex, such that the union is a closed topological disk.*

Note that although we have defined a level-z polyhex above, it still remains to be shown that such level-z polyhexes actually exist.

Appendix A.2. Existence of level-z polyhexes

We are now ready to show the existence of level-z polyhexes for all $z \in \mathbb{N}$.

Theorem 8. *For each $z \in \mathbb{N}$, the set of prototiles specified by level-z polyhexes (from Definition 5) is non-empty.*

Proof. The proof is an induction on z.

Base Case: $z = 1$. A level-1 polyhex is just a regular hexagon. It is well known that a regular hexagon satisfies the translation criterion and admits a hexagonal tiling of the Euclidean plane by a lattice of translations. Thus, the base case is established.

Inductive Hypothesis: $z = k$. Assume that level-k polyhex exists. By the definition, a level-k polyhex is a translational patch, so by the translation criterion it follows that a level-k polyhex admits a hexagonal tiling of the Euclidean plane.

Inductive Step: $z = k+1$. Take a level-k polyhex labeled (say) $\mathcal{H}_1$. From the inductive hypothesis and Theorem 7, we know that the boundary of $\mathcal{H}_1$ consists of six segments, labeled clockwise A, B, C, D, E, and F, such that each of the three pairs A-D, B-E, and C-F are translations of each other.

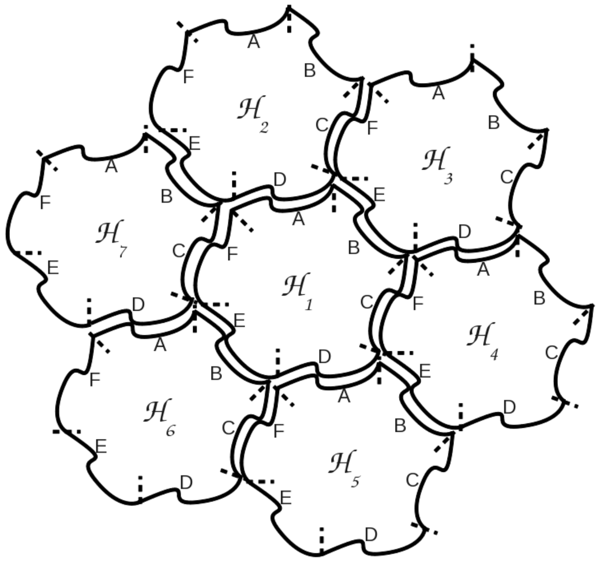

Figure A.4: Arranging 7 level-k polyhexes $\mathcal{H}_1$–$\mathcal{H}_7$ as described in the inductive step of the proof for Theorem 8.

Replicate $\mathcal{H}_1$ six times (including the labeling of the boundary segments) so that there are 7 level-k polyhexes, $\mathcal{H}_1$–$\mathcal{H}_7$, that can be translated to each other. Each level-k polyhex has six boundary segments, labeled clockwise A, B, C, D, E, and F, such that each of the three pairs A-D, B-E, and C-F are translations of each other. Denoting a segment α of a polyhex $\mathcal{H}_i$ as $\mathcal{H}_i.\alpha$, we know that $\forall i, j \in \{1..7\}, \forall \alpha \in \{A..F\}$, $H_i.\alpha$ can be translated to $H_j.\alpha$. Also, if α is A, B, C, D, E, or F, respectively, then let $\overline{\alpha}$ denote D, E, F, A, B, or C, respectively. It follows that $\forall i, j \in \{1..7\}, \forall \alpha \in \{A..F\}$, $H_i.\alpha$ can be translated to $H_j.\overline{\alpha}$

Arrange $\mathcal{H}_1$–$\mathcal{H}_7$ as follows: let $\mathcal{H}_2.D$ line up with $\mathcal{H}_1.A$, let $\mathcal{H}_3.E$ line up with $\mathcal{H}_1.B$, let $\mathcal{H}_4.F$ line up with $\mathcal{H}_1.C$, let $\mathcal{H}_5.A$ line up with $\mathcal{H}_1.D$, let $\mathcal{H}_6.B$ line up with $\mathcal{H}_1.E$, and let $\mathcal{H}_7.C$ line up with $\mathcal{H}_1.F$. Such an arrangement is illustrated in Figure A.4.[10]

Note that (a) each $\mathcal{H}_1.\alpha$ is lined up with a distinct $\mathcal{H}_i.\overline{\alpha}$, (b) $\mathcal{H}_1.\alpha$ can be translated to $\mathcal{H}_i.\overline{\alpha}$, and (c) a level-$k$ polyhex admits a hexagonal tiling of the Euclidean plane by a lattice of translations. From the above observations, it follows that the arrangement of $\mathcal{H}_1$–$\mathcal{H}_7$ is a *patch* of level-k polyhexes.

Such a construction of polyhexes satisfies parts (1) and (2) of the definition of a level-z polyhex because there are seven level-k polyhexes; $\mathcal{H}_1$ is the internal polyhex, and each other polyhex is connected to the internal polyhex ($\mathcal{H}_1$) and

[10]Note that in such an arrangement $\mathcal{H}_{i+2}.\alpha$ lines up with $\mathcal{H}_{((i+1) \mod 6)+2}.\overline{\alpha}$.

to two other external polyhexes ($\mathcal{H}_2 - \mathcal{H}_7$).

We will now determine the sides of the polyhex formed by $\mathcal{H}_1$–$\mathcal{H}_7$. Given the arrangement of $\mathcal{H}_1$–$\mathcal{H}_7$ (and that $\mathcal{H}_1$–$\mathcal{H}_7$ is a patch), it follows that $\mathcal{H}_2.C$ lines up with $\mathcal{H}_3.F$, $\mathcal{H}_3.D$ lines up with $\mathcal{H}_4.A$, $\mathcal{H}_4.E$ lines up with $\mathcal{H}_5.B$, $\mathcal{H}_5.F$ lines with with $\mathcal{H}_6.C$, $\mathcal{H}_6.A$ lines up with $\mathcal{H}_7.D$, and $\mathcal{H}_7.B$ lines up with $\mathcal{H}_2.E$ (refer to Figure A.4 as an illustrative guide). Therefore, the boundary of $\mathcal{H}_1$–$\mathcal{H}_7$ consists of the following contiguous segments: $\mathcal{H}_2.A$, $\mathcal{H}_2.B$, $\mathcal{H}_3.A$, $\mathcal{H}_3.B$, $\mathcal{H}_3.C$, $\mathcal{H}_4.B$, $\mathcal{H}_4.C$, $\mathcal{H}_4.D$, $\mathcal{H}_5.C$, $\mathcal{H}_5.D$, $\mathcal{H}_5.E$, $\mathcal{H}_6.D$, $\mathcal{H}_6.E$, $\mathcal{H}_6.F$, $\mathcal{H}_7.E$, $\mathcal{H}_7.F$, $\mathcal{H}_7.A$, and $\mathcal{H}_2.F$ (as illustrated in Figure A.4).

Consider the following six segments of the boundary of $\mathcal{H}_1$–$\mathcal{H}_7$: $\mathcal{A} = (\mathcal{H}_2.A, \mathcal{H}_2.B, \mathcal{H}_3.A)$, $\mathcal{B} = (\mathcal{H}_3.B, \mathcal{H}_3.C, \mathcal{H}_4.B)$, $\mathcal{C} = (\mathcal{H}_4.C, \mathcal{H}_4.D, \mathcal{H}_5.C)$, $\mathcal{D} = (\mathcal{H}_5.D, \mathcal{H}_5.E, \mathcal{H}_6.D)$, $\mathcal{E} = (\mathcal{H}_6.E, \mathcal{H}_6.F, \mathcal{H}_7.E)$, and $\mathcal{F} = (\mathcal{H}_7.F, \mathcal{H}_7.A, \mathcal{H}_2.F)$, where (α, β, γ) denotes the concatenation of the segments α, β, and γ.

Consider the hexagons associated with each of the segments constructed above. That is, the hexagons that contribute at least one line segment to the segments constructed above. It can be verified that these hexagons satisfy the requirements for a side of a level-$(k+1)$ polyhex in part (3) of the definition. The construction above has six sides, and each side consists of two sides of a level-k polyhex, and one side from another level-k polyhex.

To complete the proof, it remains to be shown that the polyhex formed by $\mathcal{H}_1$–$\mathcal{H}_7$ is a trans-polyhex. Note that $\mathcal{H}_2.A$ can be translated to $\mathcal{H}_5.D$, $\mathcal{H}_2.B$ can be translated to $\mathcal{H}_5.E$, and $\mathcal{H}_3.A$ can be translated to $\mathcal{H}_6.D$. Therefore, it follows that $\mathcal{A}$ and $\mathcal{D}$ are translations of each other. Similarly, it can be verified that $\mathcal{B}$ and $\mathcal{E}$ are translations of each other, and $\mathcal{C}$ and $\mathcal{F}$ are translations of each other. By applying Theorem 7 to $\mathcal{H}_1$–$\mathcal{H}_7$ we conclude that $\mathcal{H}_1$–$\mathcal{H}_7$ is a *translational patch*. Also, since each $\mathcal{H}_i$ is a polyhex, $\mathcal{H}_1$–$\mathcal{H}_7$ is a trans-polyhex.

Therefore, by construction, it follows that $\mathcal{H}_1$–$\mathcal{H}_7$ is a level-$(k+1)$ polyhex.

Thus, by induction on z, we have shown that for all $z \in \mathbb{N}$, there exists a level-z polyhex; that is, for each $z \in \mathbb{N}$, the set of prototiles specified by level-z polyhexes (from definition 5) is non-empty. □

Figure 1(a) shows a level-2 polyhex; Figure 1(b) shows how 7 level-2 polyhexes can be arranged to form a level-3 polyhex.

Appendix A.3. Sides of a level-z polyhex

Recall that a level-z polyhex has six sides. Note that each side of a polyhex is a union of hexgaons. For convenience, we denote the number of hexagons in a side as its *length*. Next, we determine the length of a level-z polyhex.

Theorem 9. *The length of a side of a level-z polyhex is:* $length(z) = 1 + \sum_{i=0}^{z-2} 3^i$.

Proof. The proof is by induction on z.

The base case is $z = 1$. By definition, each side of a level-1 polyhex consists of the single hexagon that constitutes the polyhex. It is easy to see that by substituting $z = 1$, we get $length(1) = 1 + 0 = 1$.

Suppose the side of a level-k polyhex is $length(k) = 1 + \sum_{i=0}^{k-2} 3^i$. We now show that the property is true for a level-$(k+1)$ polyhex. From the definition of the side of a level-z polyhex (in Definition 5) we know that the side of a level-$(k+1)$ polyhex consists of two sides of one level-k polyhex, and one side of another level-k polyhex. Since two of the sides come from the same level-k polyhex, they share a common hexagon (the 'meeting point' of these two sides is the common hexagon). Therefore, $length(k+1) = 3 \cdot length(k) - 1 = 3 \cdot (1 + \sum_{i=0}^{k-2} 3^i) - 1 = \sum_{i=1}^{k-1} 3^i + 2 = 1 + \sum_{i=0}^{k-1} 3^i$. □